

The holomorphic symplectic structures on hyper-Kähler manifolds of type A_∞

Kota Hattori

*Graduate School of Mathematical Sciences, University of Tokyo
3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan
khattori@ms.u-tokyo.ac.jp*

Abstract

Hyper-Kähler manifolds of type A_∞ are noncompact complete Ricci-flat Kähler manifolds of complex dimension 2, constructed by Anderson, Kronheimer, LeBrun [1] and Goto [4]. We study the holomorphic symplectic structures preserved by the natural \mathbb{C}^\times -actions on these manifolds, then show the sufficient and necessary conditions for the existence of \mathbb{C}^\times -equivariant biholomorphisms between two hyper-Kähler manifolds of type A_∞ preserving their holomorphic symplectic structures. As a consequence, we can show the existence of a complex manifold of dimension 2 on which there is a continuous family of complete Ricci-flat Kähler metrics with distinct volume growth.

1 Introduction

Hyper-Kähler manifolds of type A_∞ were first constructed by Anderson, Kronheimer and LeBrun in [1], as the first example of complete Ricci-flat Kähler manifolds with infinite topological type. Here, infinite topological type means that their homology groups are infinitely generated. After [1], Goto [4, 5] has succeeded in constructing these manifolds in another way, using hyper-Kähler quotient construction. He also constructed the higher dimensional complete hyper-Kähler manifolds with infinite topological type. Some of the topological and geometric properties of hyper-Kähler manifolds of type A_∞ were studied well in the above papers, and the author studied the volume growth of the Riemannian metrics in [6]. Then this paper focuses on the complex geometry on the hyper-Kähler manifolds of type A_∞ .

A hyper-Kähler manifold is, by definition, a Riemannian manifold (X, g) of real dimension $4n$ equipped with three complex structures I_1, I_2, I_3 satisfying the quaternionic relations $I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -\text{id}$ with respect to all of which the metric g is Kählerian. Then the holonomy group of g is a subgroup of $Sp(n)$ and g is Ricci-flat. By the definition, the hyper-Kähler manifold carry three Kähler forms defined by $\omega_i := g(I_i \cdot, \cdot)$ for $i = 1, 2, 3$, then we have an non-degenerate closed $(2, 0)$ -form $\omega_{\mathbb{C}} = \omega_3 - \sqrt{-1}\omega_2$, which is called a holomorphic symplectic form, if (X, I_1) is regarded as a complex manifold. Conversely, it is known that (g, I_1, I_2, I_3) are reconstructed from $(\omega_1, \omega_2, \omega_3)$, thus we call $\omega = (\omega_1, \omega_2, \omega_3)$ the hyper-Kähler structure over X . In this paper we regard the hyper-Kähler manifold (X, ω) as a complex manifold with respect to the complex structure I_1 .

In [6], the author computed the volume growth of hyper-Kähler manifolds of type A_{∞} . Here, the volume growth of a Riemannian manifold (X, g) is the asymptotic behavior of the function $V_g(p_0, r)$, which is defined as the volume of the geodesic ball of radius r centered at $p_0 \in X$. Then the following result was obtained.

Theorem 1.1 ([6]). *There exists a C^{∞} manifold X of $\dim_{\mathbb{R}} X = 4$ and a family of hyper-Kähler structures $\omega^{(\alpha)}$ on X for $3 < \alpha < 4$ which carry complete hyper-Kähler metrics $g^{(\alpha)}$ with*

$$0 < \liminf_{r \rightarrow +\infty} \frac{V_{g^{(\alpha)}}(p_0, r)}{r^{\alpha}} \leq \limsup_{r \rightarrow +\infty} \frac{V_{g^{(\alpha)}}(p_0, r)}{r^{\alpha}} < +\infty.$$

for any $p_0 \in X$.

The hyper-Kähler manifolds $(X, \omega^{(\alpha)})$ in the above theorem are already constructed in [1][4], and the essential part of Theorem 1.1 is the computation of the volume growth of these manifolds. In this paper we study the holomorphic symplectic structure $\omega_{\mathbb{C}}^{(\alpha)}$ over X . The period of $(X, \omega_{\mathbb{C}}^{(\alpha)})$, that is the cohomology class determined by $\omega_{\mathbb{C}}^{(\alpha)}$, is independent of α . Then the holomorphic symplectic structures $\omega_{\mathbb{C}}^{(\alpha)}$ are expected to be independent of α . We actually obtain the following result.

Theorem 1.2. *There exists a complex manifold X of $\dim_{\mathbb{C}} X = 2$ and a family of complete Ricci-flat Kähler metrics $g^{(\alpha)}$ on X for $3 < \alpha < 4$ with*

$$0 < \liminf_{r \rightarrow +\infty} \frac{V_{g^{(\alpha)}}(p_0, r)}{r^{\alpha}} \leq \limsup_{r \rightarrow +\infty} \frac{V_{g^{(\alpha)}}(p_0, r)}{r^{\alpha}} < +\infty.$$

for any $p_0 \in X$.

Theorem 1.2 will be proved as a corollary of the results in [6] and under-mentioned Theorem 1.3, which is the main result in this paper.

It is known that there are complete Ricci-flat Kähler metrics over \mathbb{C}^n who do not have the Euclidean volume growth [11]. For example, the Taub-NUT metrics over \mathbb{C}^2 are the complete Ricci-flat Kähler metrics whose volume growth are r^3 . On the other hand, Theorem 1.2 asserts the existence of a complex manifold who has a continuous family of complete Ricci-flat Kähler metrics, whose volume growth also change continuously.

To show that two hyper-Kähler quotients are biholomorphic or not, it is useful to see the GIT quotient construction and study the period. For example, Konno [9] has studied the period of holomorphic symplectic structures of toric hyper-Kähler manifolds, that are typical examples of hyper-Kähler quotients, using GIT quotient construction. However, this method is not enough for studying the case of hyper-Kähler manifolds of type A_∞ , because these manifolds are obtained by taking quotients by the action of infinite dimensional Lie groups on infinite dimensional manifolds, then we should develop other methods to show that $(X, \omega_{\mathbb{C}}^{(\alpha_1)})$ and $(X, \omega_{\mathbb{C}}^{(\alpha_2)})$ are biholomorphic.

In this paper, we consider when two hyper-Kähler manifolds of type A_∞ become isomorphic as holomorphic symplectic manifolds. Let (X_i, ω_i) be the hyper-Kähler manifolds of type A_∞ for $i = 0, 1$. Then there are the natural \mathbb{C}^\times -actions over X_i preserving their holomorphic symplectic structures $\omega_{i, \mathbb{C}}$, and the complex moment maps $\mu_{i, \mathbb{C}} : X_i \rightarrow \mathbb{C}$. Since the complex moment maps are \mathbb{C}^\times -invariant, they define complex valued continuous functions $[\mu_{i, \mathbb{C}}] : X_i/\mathbb{C}^\times \rightarrow \mathbb{C}$ on the quotient topological spaces X_i/\mathbb{C}^\times . Moreover, the \mathbb{C}^\times -actions define natural partial order structures on the quotient spaces X_i/\mathbb{C}^\times . Then we obtain the following result.

Theorem 1.3. *There exists a \mathbb{C}^\times -equivariant biholomorphic map $f : X_0 \rightarrow X_1$ with $f^*\omega_{1, \mathbb{C}} = \omega_{0, \mathbb{C}}$ if and only if there is a homeomorphism $\mathbf{h} : X_0/\mathbb{C}^\times \rightarrow X_1/\mathbb{C}^\times$ preserving the order structures and $[\mu_{1, \mathbb{C}}] \circ \mathbf{h} - [\mu_{0, \mathbb{C}}]$ is constant.*

The above theorem is proven as follows in this paper. Put

$$X_i^* := X_i \setminus \{p \in X_i; pg = p \text{ for all } g \in \mathbb{C}^\times\}.$$

Then we have an open covering $X_i^* = \bigcup_s X_i^s$ where each X_i^s is biholomorphic to $\mathbb{C}^\times \times \mathbb{C}$, consequently biholomorphic maps $X_0^s \rightarrow X_1^s$ are obtained. Moreover we can show that these biholomorphic maps glue on the intersections, therefore a biholomorphic map $X_0^* \rightarrow X_1^*$ is obtained, and it extends to the biholomorphic map $X_0 \rightarrow X_1$ by Hartogs' extension theorem. The similar way is used already in [11] to show that the complex structure over \mathbb{R}^4 given by the Taub-NUT metric is biholomorphic to \mathbb{C}^2 . But in our case,

the topological structure of X_i/\mathbb{C}^\times is so complicated that we should study them precisely.

This paper is organized as follows. First of all we review the results obtained in [4] in Sections 2 and 3. In Section 2, we construct hyper-Kähler manifolds of type A_∞ by using hyper-Kähler quotient constructions and see that there exists a natural S^1 -action. In Section 3, we review the another quotient construction of hyper-Kähler manifolds of type A_∞ , and see that the manifolds obtained in Sections 2 and 3 are isomorphic as the holomorphic symplectic manifolds.

In Section 4, we study the topological properties of topological quotient spaces obtained from hyper-Kähler manifolds of type A_∞ by taking the quotient by \mathbb{C}^\times -action. We can also see that there exist natural partial order structures.

In Section 5 we construct biholomorphisms between two hyper-Kähler manifolds of type A_∞ , which satisfy the assumption of Theorem 1.3. As a consequence, we apply Theorem 1.3 for more concrete case, and obtain Theorem 1.2 and other results in Section 6.

2 Hyper-Kähler manifolds of type A_∞

2.1 Hyper-Kähler quotient construction

In this section, we review shortly the construction of hyper-Kähler manifolds of type A_∞ along [4]. Although they can be constructed by Gibbons-Hawking ansatz [1], we need hyper-Kähler quotient construction in [4] for arguments in Section 4. For more details of construction and basic facts, see [1][4] or review in Section 2 of [6].

First of all, we describe the definition of hyper-Kähler manifolds.

Definition 2.1. Let (X, g) be a Riemannian manifold of dimension $4n$, I_1, I_2, I_3 be integrable complex structures on X , and g is a hermitian metric with respect to each I_i . Then (X, g, I_1, I_2, I_3) is a hyper-Kähler manifold if (I_1, I_2, I_3) satisfies the relations $I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -1$ and each fundamental 2-form $\omega_i := g(I_i \cdot, \cdot)$, that is, (X, g, I_i) is kählerian.

Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k = \mathbb{C} \oplus \mathbb{C}j$ be quaternion and $\text{Im}\mathbb{H} = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be its Imaginary part. Then an $\text{Im}\mathbb{H}$ -valued 2-form $\omega := i\omega_1 + j\omega_2 + k\omega_3 \in \Omega^2(X) \otimes \text{Im}\mathbb{H}$ are constructed from the hyper-Kähler structure (g, I_1, I_2, I_3) . Conversely, (g, I_1, I_2, I_3) is reconstructed from ω . Hence we call ω the hyper-Kähler structure on X instead of (g, I_1, I_2, I_3) .

To construct hyper-Kähler manifolds of type A_∞ , we prepare an infinite countable set \mathbf{I} and a parameter space

$$(\mathrm{Im}\mathbb{H})_0^{\mathbf{I}} := \{\lambda = (\lambda_n)_{n \in \mathbf{I}} \in (\mathrm{Im}\mathbb{H})^{\mathbf{I}}; \sum_{n \in \mathbf{I}} \frac{1}{1 + |\lambda_n|} < +\infty\}.$$

For a set S , we denote by $S^{\mathbf{I}}$ the set of all maps from \mathbf{I} to S . An element of $x \in S^{\mathbf{I}}$ is written as $x = (x_n)_{n \in \mathbf{I}}$. Then we have a Hilbert space

$$M_{\mathbf{I}} := \{v \in \mathbb{H}^{\mathbf{I}}; \|v\|_{\mathbf{I}}^2 < +\infty\},$$

where

$$\langle u, v \rangle_{\mathbf{I}} := \sum_{n \in \mathbf{I}} u_n \bar{v}_n, \quad \|v\|_{\mathbf{I}}^2 := \langle v, v \rangle_{\mathbf{I}}$$

for $u, v \in \mathbb{H}^{\mathbf{I}}$. Here, $\bar{v}_n \in \mathbb{H}$ is the quaternionic conjugate of v_n defined by $\overline{a + bi + cj + dk} := a - bi - cj - dk$ for $a, b, c, d \in \mathbb{R}$.

Now we fix $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbf{I}}$, and take $\Lambda \in \mathbb{H}^{\mathbf{I}}$ to be $\Lambda_n i \bar{\Lambda}_n = \lambda_n$. Then we have the following Hilbert manifolds

$$\begin{aligned} M_{\Lambda} &:= \Lambda + M_{\mathbf{I}} = \{\Lambda + v; v \in M_{\mathbf{I}}\}, \\ G_{\lambda} &:= \{g \in (S^1)^{\mathbf{I}}; \sum_{n \in \mathbf{I}} (1 + |\lambda_n|) |1 - g_n|^2 < +\infty, \prod_{n \in \mathbf{I}} g_n = 1\}, \\ \mathfrak{g}_{\lambda} &:= \mathrm{Lie}(G_{\lambda}) = \{\xi \in \mathbb{R}^{\mathbf{I}}; \sum_{n \in \mathbf{I}} (1 + |\lambda_n|) |\xi_n|^2 < +\infty, \sum_{n \in \mathbf{I}} \xi_n = 0\}. \end{aligned}$$

The convergence of $\prod_{n \in \mathbf{I}} g_n$ and $\sum_{n \in \mathbf{I}} \xi_n$ follows from the condition $\sum_{n \in \mathbf{I}} (1 + |\lambda_n|)^{-1} < +\infty$. Then G_{λ} is a Hilbert Lie group whose Lie algebra is \mathfrak{g}_{λ} . We can define a right action of G_{λ} on M_{Λ} by $xg := (x_n g_n)_{n \in \mathbf{I}}$ for $x \in M_{\Lambda}$, $g \in G_{\lambda}$. Here the product of x_n and g_n is given by regarding S^1 as the subset of \mathbb{H} by the natural injections $S^1 \subset \mathbb{C} \subset \mathbb{H}$. Then G_{λ} acts on M_{Λ} preserving the hyper-Kähler structure, and we have the hyper-Kähler moment map $\hat{\mu}_{\Lambda} : M_{\Lambda} \rightarrow \mathrm{Im}\mathbb{H} \otimes \mathfrak{g}_{\lambda}^*$ defined by

$$\langle \hat{\mu}_{\Lambda}(x), \xi \rangle := \sum_{n \in \mathbf{I}} (x_n i \bar{x}_n - \Lambda_n i \bar{\Lambda}_n) \xi_n \in \mathrm{Im}\mathbb{H}$$

for $x \in M_{\Lambda}$, $\xi \in \mathfrak{g}_{\lambda}$. If \mathbf{I} is a finite set.

Since $\hat{\mu}_{\Lambda}$ is G_{λ} -invariant, then G_{λ} acts on the inverse image

$$\hat{\mu}_{\Lambda}^{-1}(0) = \{x \in M_{\Lambda}; x_n i \bar{x}_n - \lambda_n = x_m i \bar{x}_m - \lambda_m \text{ for all } n, m \in \mathbf{I}\}$$

Hence we obtain the quotient space $\hat{\mu}_{\Lambda}^{-1}(0)/G_{\lambda}$ which is called the hyper-Kähler quotient.

Definition 2.2. An element $\lambda \in (\operatorname{Im}\mathbb{H})_0^{\mathbf{I}}$ is generic if $\lambda_n - \lambda_m \neq 0$ for all distinct $n, m \in \mathbf{I}$.

Theorem 2.3 ([4]). *If $\lambda \in (\operatorname{Im}\mathbb{H})_0^{\mathbf{I}}$ is generic, then $\hat{\mu}_\Lambda^{-1}(0)/G_\lambda$ is a smooth manifold of real dimension 4, and the hyper-Kähler structure on M_Λ induces a hyper-Kähler structure ω_λ on $\hat{\mu}_\Lambda^{-1}(0)/G_\lambda$.*

The quotient space $\hat{\mu}_\Lambda^{-1}(0)/G_\lambda$ seems to depend on the choice of $\Lambda \in \mathbb{H}^{\mathbf{I}}$, but the induced hyper-Kähler structure on $\hat{\mu}_\Lambda^{-1}(0)/G_\lambda$ depends only on λ from the argument of Section 2 of [6]. Thus we may put

$$\begin{aligned} X_{HKQ}(\lambda) &:= \hat{\mu}_\Lambda^{-1}(0)/G_\lambda \\ &= \{x \in M_\Lambda; x_n i \bar{x}_n - \lambda_n \text{ is independent of } n \in \mathbf{I}\}/G_\lambda, \end{aligned}$$

and call it hyper-Kähler manifold of type A_∞

Recall that we assume that \mathbf{I} is infinite. If $\sharp\mathbf{I} = k + 1 < +\infty$, then $(X_{HKQ}(\lambda), \omega_\lambda)$ becomes an ALE hyper-Kähler manifold of type A_k [3].

2.2 S^1 -actions and moment maps

In [4], an S^1 -action on $X_{HKQ}(\lambda)$ preserving the hyper-Kähler structure defined as follows. We denote by $[x] \in \hat{\mu}_\Lambda^{-1}(0)/G_\lambda$ the equivalence class represented by $x \in \hat{\mu}_\Lambda^{-1}(0)$. Fix $m \in \mathbf{I}$ and put

$$[x]g := [x_m g, (x_n)_{n \in \mathbf{I} \setminus \{m\}}]$$

for $x = (x_m, (x_n)_{n \in \mathbf{I} \setminus \{m\}}) \in \hat{\mu}_\Lambda^{-1}(0)$ and $g \in S^1$. This definition is independent of the choice of $m \in \mathbf{I}$, and we have the action of S^1 on $X_{HKQ}(\lambda)$. The hyper-Kähler moment map $\mu_\lambda : X_{HKQ}(\lambda) \rightarrow \operatorname{Im}\mathbb{H} = \mathbb{R}^3$ is defined by

$$\mu_\lambda([x]) := x_n i \bar{x}_n - \lambda_n \in \operatorname{Im}\mathbb{H}.$$

The right hand side is independent of the choice of $n \in \mathbf{I}$ since x is an element of $\hat{\mu}_\Lambda^{-1}(0)$.

Put

$$\begin{aligned} X_{HKQ}(\lambda)^* &:= \{[x] \in X_{HKQ}(\lambda); x_n \neq 0 \text{ for all } n \in \mathbf{I}\}, \\ Y_\lambda &:= \operatorname{Im}\mathbb{H} \setminus \{-\lambda_n; n \in \mathbf{I}\}, \end{aligned}$$

then we have a principal S^1 -bundle $\mu_\lambda|_{X_{HKQ}(\lambda)^*} : X_{HKQ}(\lambda)^* \rightarrow Y_\lambda$, and S^1 acts on $X_{HKQ}(\lambda) \setminus X_{HKQ}(\lambda)^*$ trivially.

Conversely, on the total spaces of some principal S^1 -bundle over Y_λ , hyper-Kähler structures preserved by S^1 -actions are constructed in [1] by

Gibbons-Hawking ansatz. It is shown in [4] that each $X_{HKQ}(\lambda)$ is isomorphic to one of that constructed by Gibbons-Hawking ansatz.

By observing the Gibbons-Hawking construction, it is easy to see that $X_{HKQ}(\lambda)$ and $X_{HKQ}(\lambda')$ are isomorphic as hyper-Kähler manifolds if λ and λ' satisfy one of the following relations; (i) $\lambda'_n - \lambda_n \in \text{Im}\mathbb{H}$ are independent of n , (ii) $\lambda'_n = \lambda_{a(n)}$ for some bijective maps $a : \text{Im}\mathbb{H} \rightarrow \text{Im}\mathbb{H}$.

We can also show easily that $X_{HKQ}(\lambda) \cong X_{HKQ}(-\lambda)$ by constructing an isomorphism explicitly.

3 Holomorphic description

In this section we compare hyper-Kähler quotients $\hat{\mu}_\Lambda^{-1}(0)/G_\lambda$ with another kind of quotient spaces $\hat{\mu}_{\Lambda,\mathbb{C}}^{-1}(0)/G_\lambda^\mathbb{C}$ along [4], where $\hat{\mu}_{\Lambda,\mathbb{C}}$ is the complex valued component of $\hat{\mu}_\Lambda$ and $G_\lambda^\mathbb{C}$ is the complexification of G_λ .

First of all, we complexify the Hilbert Lie group G_λ as follows,

$$\begin{aligned} G_\lambda^\mathbb{C} &:= \{g \in (\mathbb{C}^\times)^\mathbf{I}; \sum_{n \in \mathbf{I}} (1 + |\lambda_n|)|1 - g_n|^2 < +\infty, \prod_{n \in \mathbf{I}} g_n = 1\}, \\ \mathfrak{g}_\lambda^\mathbb{C} &:= \mathfrak{g}_\lambda \otimes \mathbb{C} = \{\xi \in \mathbb{C}^\mathbf{I}; \sum_{n \in \mathbf{I}} (1 + |\lambda_n|)|\xi_n|^2 < +\infty, \sum_{n \in \mathbf{I}} \xi_n = 0\}, \end{aligned}$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Then $G_\lambda^\mathbb{C}$ acts smoothly on M_Λ , where $\Lambda \in \mathbb{H}^\mathbf{I}$ satisfies $\Lambda_n i \bar{\Lambda}_n = \lambda_n$.

From now on we write $\zeta = \zeta_\mathbb{R}i - \zeta_\mathbb{C}k = (\zeta_\mathbb{R}, \zeta_\mathbb{C}) \in \text{Im}\mathbb{H}$ along the decomposition $\text{Im}\mathbb{H} = \mathbb{R}i \oplus \mathbb{C}k$. Similarly, we write $\lambda = \lambda_\mathbb{R}i - \lambda_\mathbb{C}k = (\lambda_\mathbb{R}, \lambda_\mathbb{C})$ for $\lambda \in (\text{Im}\mathbb{H})^\mathbf{I}$, where $\lambda_\mathbb{R} \in \mathbb{R}^\mathbf{I}$ and $\lambda_\mathbb{C} \in \mathbb{C}^\mathbf{I}$. The hyper-Kähler moment map $\hat{\mu}_\Lambda$ is also decomposed into two components as $\hat{\mu}_\Lambda = \hat{\mu}_{\Lambda,\mathbb{R}} \cdot i - \hat{\mu}_{\Lambda,\mathbb{C}} \cdot k$. Then $\hat{\mu}_{\Lambda,\mathbb{R}} : M_\Lambda \rightarrow \mathfrak{g}_\lambda^*$ and $\hat{\mu}_{\Lambda,\mathbb{C}} : M_\Lambda \rightarrow (\mathfrak{g}_\lambda^\mathbb{C})^*$ are written as

$$\begin{aligned} \langle \hat{\mu}_{\Lambda,\mathbb{R}}(z + wj), \xi \rangle &= \sum_{n \in \mathbf{I}} (|z_n|^2 - |w_n|^2 - \lambda_{n,\mathbb{R}})\xi_n, \\ \langle \hat{\mu}_{\Lambda,\mathbb{C}}(z + wj), \eta \rangle &= \sum_{n \in \mathbf{I}} (2z_n w_n - \lambda_{n,\mathbb{C}})\eta_n, \end{aligned}$$

for $z + wj = (z_n + w_n j)_{n \in \mathbf{I}} \in M_\Lambda$, $\xi \in \mathfrak{g}_\lambda$ and $\eta \in \mathfrak{g}_\lambda^\mathbb{C}$, where $z_n, w_n \in \mathbb{C}$ and $\lambda_n = \lambda_{n,\mathbb{R}}i - \lambda_{n,\mathbb{C}}k$. Then $\hat{\mu}_{\Lambda,\mathbb{C}}$ is $G_\lambda^\mathbb{C}$ invariant.

Definition 3.1. Let $t = (t_n)_{n \in \mathbf{I}} \in \mathbb{R}^\mathbf{I}$. Then $z + wj \in M_\Lambda$ is t -stable if $|z_n|^2 + |w_m|^2 > 0$ holds for any $n, m \in \mathbf{I}$ which satisfy $t_n > t_m$.

Now we put

$$\hat{\mu}_{\Lambda,\mathbb{C}}^{-1}(0)_t := \{z + wj \in \hat{\mu}_{\Lambda,\mathbb{C}}^{-1}(0); z + wj \text{ is } t\text{-stable}\}.$$

Then $G_\lambda^\mathbb{C}$ acts on $\hat{\mu}_{\Lambda,\mathbb{C}}^{-1}(0)_t$. If the quotient space $X_{GIT}(\lambda) := \hat{\mu}_{\Lambda,\mathbb{C}}^{-1}(0)_{\lambda_\mathbb{R}}/G_\lambda^\mathbb{C}$ becomes a smooth manifold, then the standard nowherevanishing $(2, 0)$ -form $\sum_{n \in \mathbf{I}} dz_n \wedge dw_n$ over M_Λ induces a holomorphic symplectic form $\omega_{\lambda,\mathbb{C}}$ on $X_{GIT}(\lambda)$. Then $(X_{GIT}(\lambda), \omega_{\lambda,\mathbb{C}})$ depends only on λ , not depends on Λ .

Theorem 3.2 ([4]). *Let $\lambda \in (\text{Im}\mathbb{H})_0^\mathbf{I}$ be generic. Then the quotient space $X_{GIT}(\lambda)$ becomes a complex manifold of dimension 2.*

For any generic λ , $\hat{\mu}_\Lambda^{-1}(0)$ is a subset of $\hat{\mu}_{\Lambda,\mathbb{C}}^{-1}(0)_{\lambda_\mathbb{R}}$. Then this inclusion induces

$$\phi_\lambda : X_{HKQ}(\lambda) = \hat{\mu}_\Lambda^{-1}(0)/G_\lambda \rightarrow \hat{\mu}_{\Lambda,\mathbb{C}}^{-1}(0)_{\lambda_\mathbb{R}}/G_\lambda^\mathbb{C} = X_{GIT}(\lambda),$$

which is an biholomorphism preserving the holomorphic structure, namely,

$$\phi_\lambda^* \omega_{\lambda,\mathbb{C}} = \omega_{\lambda,3} - \sqrt{-1}\omega_{\lambda,2},$$

where $\omega_\lambda = \omega_{\lambda,1}i + \omega_{\lambda,2}j + \omega_{\lambda,3}k$ is the hyper-Kähler structure on $X_{HKQ}(\lambda)$. Here, $\omega_{\lambda,3} - \sqrt{-1}\omega_{\lambda,2}$ is the holomorphic symplectic structures over $X_{HKQ}(\lambda)$ with respect to the complex structure $I_{\lambda,1}$. From now on we write

$$(X(\lambda), \omega_{\lambda,\mathbb{C}}) := (X_{HKQ}(\lambda), \omega_{\lambda,3} - \sqrt{-1}\omega_{\lambda,2}) = (X_{GIT}(\lambda), \omega_{\lambda,\mathbb{C}})$$

if it is not necessary to distinguish them.

In Section 2.2, we have seen that $X(\lambda)$ has a natural S^1 -action. Then by complexifying the action, we have a holomorphic \mathbb{C}^\times -action on $X(\lambda)$ preserving $\omega_{\lambda,\mathbb{C}}$ defined by

$$[z + wj]g := [z_m g + w_m g^{-1}, (z_n + w_n)_{n \in \mathbf{I} \setminus \{m\}}].$$

It is easy to see that \mathbb{C}^\times acts freely on $X(\lambda)^* = X_{HKQ}(\lambda)^*$, and trivially on $X(\lambda) \setminus X(\lambda)^*$.

4 Topological structure of $X(\lambda)/\mathbb{C}^\times$

In the previous section, we obtain \mathbb{C}^\times -action on $X(\lambda)$. In this section we will study the topology of the quotient space $X(\lambda)/\mathbb{C}^\times$ with the quotient topology.

4.1 The topological space homeomorphic to $X(\lambda)/\mathbb{C}^\times$

First of all, we define a certain equivalence relation \sim_λ in $\text{Im}\mathbb{H}$, which depends on $\lambda \in (\text{Im}\mathbb{H})_0^\mathbf{I}$, then we show that there exists a homeomorphism from $X(\lambda)/\mathbb{C}^\times$ to $\text{Im}\mathbb{H}/\sim_\lambda$.

Put $Z_\lambda := \{-\lambda_n \in \text{Im}\mathbb{H}; n \in \mathbf{I}\}$ for $\lambda \in (\text{Im}\mathbb{H})_0^\mathbf{I}$. Then we have a disjoint union $\text{Im}\mathbb{H} = Y_\lambda \sqcup Z_\lambda$.

Definition 4.1. Let $\lambda \in (\operatorname{Im}\mathbb{H})_0^{\mathbf{I}}$ and $\eta_1, \eta_2 \in \operatorname{Im}\mathbb{H}$. We write $\eta_1 \sim_\lambda \eta_2$ if they satisfy one of the following conditions; (i) η_1 and η_2 satisfy $\eta_{1,\mathbb{C}} = \eta_{2,\mathbb{C}}$ and $t\eta_1 + (1-t)\eta_2 \in Y_\lambda$ for all $t \in [0, 1]$, (ii) $\eta_1 = \eta_2 \in Z_\lambda$.

Now we obtain quotient spaces $X(\lambda)/\mathbb{C}^\times$ and $\operatorname{Im}\mathbb{H}/\sim_\lambda$ with quotient topology. Next we construct a homeomorphism between them.

Let $\mu_\lambda : X(\lambda) \rightarrow \operatorname{Im}\mathbb{H}$ be the hyper-Kähler moment map defined in Section 2. We will show that μ_λ induces a continuous map from $X(\lambda)/\mathbb{C}^\times$ to $\operatorname{Im}\mathbb{H}/\sim_\lambda$ by using the following lemma.

Lemma 4.2. Let $[z+wj] \in X(\lambda)^*$ and $g \in \mathbb{C}^\times$. Then we have $\mu_\lambda([z+wj]) \sim_\lambda \mu_\lambda([z+wj]g)$ and

$$\log |g|^2 = \int_{\mu_{\lambda,\mathbb{R}}([z+wj])}^{\mu_{\lambda,\mathbb{R}}([z+wj]g)} \Phi_\lambda(t, \zeta_{\mathbb{C}}) dt,$$

where Φ_λ is defined by

$$\Phi_\lambda(\zeta) := \frac{1}{4} \sum_{n \in \mathbf{I}} \frac{1}{|\zeta + \lambda_n|}$$

for $\zeta \in Y_\lambda$.

Proof. Take $\tilde{g} = (\tilde{g}_n)_{n \in \mathbf{I}} \in (\mathbb{C}^\times)^{\mathbf{I}}$ to be $\sum_{n \in \mathbf{I}} |1 - \tilde{g}_n|^2 < \infty$ and $g = \prod_{n \in \mathbf{I}} \tilde{g}_n$. Now we regard $z+wj$ as an element of $\hat{\mu}_\Lambda^{-1}(0)$, and suppose $(z_n \tilde{g}_n + w_n \tilde{g}_n^{-1} j) \in \hat{\mu}_\Lambda^{-1}(0)$.

Put $\zeta = \mu_\lambda([z+wj])$ and $\eta = \mu_\lambda([z+wj]g)$. Then we have

$$\begin{aligned} |z_n|^2 - |w_n|^2 &= \lambda_{n,\mathbb{R}} + \zeta_{\mathbb{R}}, & 2z_n w_n &= \lambda_{n,\mathbb{C}} + \zeta_{\mathbb{C}}, \\ |z_n \tilde{g}_n|^2 - |w_n \tilde{g}_n^{-1}|^2 &= \lambda_{n,\mathbb{R}} + \eta_{\mathbb{R}}, & 2z_n w_n &= \lambda_{n,\mathbb{C}} + \eta_{\mathbb{C}}, \end{aligned}$$

accordingly we have $\zeta_{\mathbb{C}} = \eta_{\mathbb{C}}$. Since $[z+wj] \in X(\lambda)^*$, we may suppose $|z_n|^2 + |w_n|^2 \neq 0$ for all $n \in \mathbf{I}$. Then \tilde{g}_n satisfies

$$\begin{aligned} |\tilde{g}_n|^2 &= \frac{|\eta + \lambda_n| + \eta_{\mathbb{R}} + \lambda_{n,\mathbb{R}}}{|\zeta + \lambda_n| + \zeta_{\mathbb{R}} + \lambda_{n,\mathbb{R}}} \quad (\text{if } z_n \neq 0), \\ |\tilde{g}_n|^{-2} &= \frac{|\eta + \lambda_n| - (\eta_{\mathbb{R}} + \lambda_{n,\mathbb{R}})}{|\zeta + \lambda_n| - (\zeta_{\mathbb{R}} + \lambda_{n,\mathbb{R}})} \quad (\text{if } w_n \neq 0). \end{aligned}$$

Now we put $\mathbf{I}_\pm(\zeta) := \{n \in \mathbf{I}; \pm(\zeta_{\mathbb{R}} + \lambda_{n,\mathbb{R}}) > 0\}$. Since $|\tilde{g}_n|^2$ and $|\tilde{g}_n|^{-2}$ should be positive, we have $\eta = (\eta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \sim_\lambda \zeta$. Then we obtain

$$\begin{aligned} F_\lambda(\eta_{\mathbb{R}}, \zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) := \log |g|^2 &= \sum_{n \in \mathbf{I}_+(\zeta)} \log \frac{|\eta + \lambda_n| + \eta_{\mathbb{R}} + \lambda_{n,\mathbb{R}}}{|\zeta + \lambda_n| + \zeta_{\mathbb{R}} + \lambda_{n,\mathbb{R}}} \\ &\quad + \sum_{n \in \mathbf{I}_-(\zeta)} \log \frac{|\zeta + \lambda_n| - (\zeta_{\mathbb{R}} + \lambda_{n,\mathbb{R}})}{|\eta + \lambda_n| - (\eta_{\mathbb{R}} + \lambda_{n,\mathbb{R}})}, \end{aligned}$$

where $\eta = \eta_{\mathbb{R}}i - \zeta_{\mathbb{C}}k$, then we have $\log |g|^2 = F_{\lambda}(\eta_{\mathbb{R}}, \zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$. The function F_{λ} is smooth at $(\eta_{\mathbb{R}}, \zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$ if $\eta, \zeta \in Y_{\lambda}$. Then we have

$$\frac{\partial F_{\lambda}}{\partial \eta_{\mathbb{R}}} = \Phi_{\lambda}(\eta_{\mathbb{R}}, \zeta_{\mathbb{C}}) > 0.$$

Since $F_{\lambda}(\zeta_{\mathbb{R}}, \zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) = 0$, we obtain

$$\log |g|^2 = \int_{\zeta_{\mathbb{R}}}^{\eta_{\mathbb{R}}} \Phi_{\lambda}(t, \zeta_{\mathbb{C}}) dt.$$

□

It is obvious that $[z + wj] = [z + wj]g$ if $\mu_{\lambda}([z + wj]) \in Z_{\lambda}$. Then the hyper-Kähler moment map μ_{λ} induces $[\mu_{\lambda}] : X(\lambda)/\mathbb{C}^{\times} \rightarrow \text{Im}\mathbb{H}/\sim_{\lambda}$ from Lemma 4.2. Since μ_{λ} is continuous and surjective, $[\mu_{\lambda}]$ is also continuous and surjective.

Proposition 4.3. *Let $\lambda \in (\text{Im}\mathbb{H})_0^{\text{I}}$ be generic. Then $[\mu_{\lambda}] : X(\lambda)/\mathbb{C}^{\times} \rightarrow \text{Im}\mathbb{H}/\sim_{\lambda}$ is a homeomorphism.*

Proof. It suffices to show that $[\mu_{\lambda}]$ is an injective and open map.

Let $[z + wj], [z' + w'j] \in X(\lambda)$ satisfy $\mu_{\lambda}([z + wj]) \sim_{\lambda} \mu_{\lambda}([z' + w'j])$. If $\mu_{\lambda}([z + wj]) \in Z_{\lambda}$, then $[z + wj] = [z' + w'j]$. If $\mu_{\lambda}([z + wj]) \in Y_{\lambda}$, then $\mu_{\lambda}([z' + w'j])$ is also an element of Y_{λ} . If we take $g \in \mathbb{C}^{\times}$ to be

$$\log |g|^2 = \int_{\mu_{\lambda, \mathbb{R}}([z + wj])}^{\mu_{\lambda, \mathbb{R}}([z' + w'j])} \Phi_{\lambda}(t, \zeta_{\mathbb{C}}) dt,$$

then we have $\mu_{\lambda}([z + wj]g) = \mu_{\lambda}([z' + w'j])$. Since S^1 acts on $\mu_{\lambda}^{-1}(\zeta)$ transitively for all $\zeta \in \text{Im}\mathbb{H}$, there exists $\sigma \in S^1$ such that $[z + wj]g\sigma = [z' + w'j]$. Thus the injectivity has been proven.

The openness of $[\mu_{\lambda}]$ is easily shown by the elementary argument of general topology. □

From now on we identify $X(\lambda)/\mathbb{C}^{\times}$ with $\text{Im}\mathbb{H}/\sim_{\lambda}$ by $[\mu_{\lambda}]$. To study the topological properties of $X(\lambda)/\mathbb{C}^{\times}$, we often observe $\text{Im}\mathbb{H}/\sim_{\lambda}$ for convenience.

Now let $p_{\lambda} : X(\lambda) \rightarrow X(\lambda)/\mathbb{C}^{\times}$ and $\pi_{\lambda} : \text{Im}\mathbb{H} \rightarrow \text{Im}\mathbb{H}/\sim_{\lambda}$ be the quotient maps. Then $\mu_{\lambda, \mathbb{C}} : X(\lambda) \rightarrow \mathbb{C}$ induces a continuous map $[\mu_{\lambda, \mathbb{C}}] : X(\lambda)/\mathbb{C}^{\times} \rightarrow \mathbb{C}$ satisfying $[\mu_{\lambda, \mathbb{C}}] \circ p_{\lambda} = \mu_{\lambda, \mathbb{C}}$. On the other hand, the orthogonal projection $\text{pr}_{\mathbb{C}} : \text{Im}\mathbb{H} \rightarrow \mathbb{C}$ defined by $\text{pr}_{\mathbb{C}}(\zeta) := \zeta_{\mathbb{C}}$ induces a continuous map $[\text{pr}_{\mathbb{C}}]_{\lambda} : \text{Im}\mathbb{H}/\sim_{\lambda} \rightarrow \mathbb{C}$ by $[\text{pr}_{\mathbb{C}}]_{\lambda} \circ \pi_{\lambda} = \text{pr}_{\mathbb{C}}$. Note that $[\mu_{\lambda, \mathbb{C}}]$ is identified with $[\text{pr}_{\mathbb{C}}]_{\lambda}$ by $[\mu_{\lambda}]$, that is, $[\mu_{\lambda, \mathbb{C}}] = [\text{pr}_{\mathbb{C}}]_{\lambda} \circ [\mu_{\lambda}]$.

There exists a natural partial order in $\text{Im}\mathbb{H}/\sim_{\lambda}$ defined as follows.

Definition 4.4. For $\zeta, \eta \in \text{Im}\mathbb{H}$, we write $\pi_\lambda(\zeta) \prec \pi_\lambda(\eta)$ if $\zeta_{\mathbb{C}} = \eta_{\mathbb{C}}$ and $\zeta_{\mathbb{R}} < \eta_{\mathbb{R}}$. Moreover we write $\pi_\lambda(\zeta) \preceq \pi_\lambda(\eta)$ if $\pi_\lambda(\zeta) \prec \pi_\lambda(\eta)$ or $\pi_\lambda(\zeta) = \pi_\lambda(\eta)$.

The above definition is well-defined and we have the structure of partially ordered set on $\text{Im}\mathbb{H}/\sim_\lambda$.

4.2 The topological structures of $\text{Im}\mathbb{H}/\sim_\lambda$

In this subsection we fix arbitrary generic $\lambda \in (\text{Im}\mathbb{H})_0^{\mathbf{I}}$.

For an open set $V \subset \mathbb{C}$, put $\pi_\lambda(Y_\lambda)|_V := [\text{pr}_{\mathbb{C}}]_\lambda^{-1}(V) \cap \pi_\lambda(Y_\lambda)$ and

$$\Gamma(V, \pi_\lambda(Y_\lambda)|_V) := \{s : V \rightarrow \pi_\lambda(Y_\lambda)|_V; s \text{ is continuous, } [\text{pr}_{\mathbb{C}}]_\lambda \circ s = \text{id}_V\}.$$

Here, the topology of \mathbb{C} is the ordinary one as Euclidean space. Under the identification $\text{Im}\mathbb{H} = \mathbb{R} \times \mathbb{C}$ by $\zeta = \zeta_{\mathbb{R}}i - \zeta_{\mathbb{C}}k = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$, all $s \in \Gamma(V, \pi_\lambda(Y_\lambda)|_V)$ are written as $s(z) = \pi_\lambda(\tilde{s}(z), z)$ for some continuous function $\tilde{s} : V \rightarrow \mathbb{R}$ such that the graph of \tilde{s} does not intersect Y_λ .

Let $s_1, s_2 : \mathbb{C} \rightarrow \pi_\lambda(Y_\lambda)$ satisfy $[\text{pr}_{\mathbb{C}}]_\lambda \circ s_1 = [\text{pr}_{\mathbb{C}}]_\lambda \circ s_2 = \text{id}_{\mathbb{C}}$, but are not necessary to be continuous, and put

$$\begin{aligned} \mathbf{I}_\lambda^+(s_i) &:= \{n \in \mathbf{I}; \pi_\lambda(-\lambda_n) \prec s_i(-\lambda_{n,\mathbb{C}})\}, \\ \mathbf{I}_\lambda^-(s_i) &:= \{n \in \mathbf{I}; \pi_\lambda(-\lambda_n) \succ s_i(-\lambda_{n,\mathbb{C}})\}. \end{aligned}$$

Then we have a disjoint union $\mathbf{I} = \mathbf{I}_\lambda^+(s_i) \sqcup \mathbf{I}_\lambda^-(s_i)$. Then we define a map $k_{s_1, s_2} : \mathbb{C} \rightarrow \mathbb{Z}$ by

$$k_{s_1, s_2}(z) := \sharp(\mathbf{I}_z \cap \mathbf{I}_\lambda^+(s_2) \cap \mathbf{I}_\lambda^-(s_1)) - \sharp(\mathbf{I}_z \cap \mathbf{I}_\lambda^+(s_1) \cap \mathbf{I}_\lambda^-(s_2))$$

for $z \in \mathbb{C}$, where $\mathbf{I}_z := \{n \in \mathbf{I}; -\lambda_{n,\mathbb{C}} = z\}$. If s_1, s_2 are described as $s_i(z) = (\tilde{s}_i(z), z)$ for some $\tilde{s}_i : \mathbb{C} \rightarrow \mathbb{R}$, we may write

$$\begin{aligned} k_{s_1, s_2}(z) &= \sharp\{n \in \mathbf{I}; -\lambda_{n,\mathbb{C}} = z, \tilde{s}_1(z) < -\lambda_{n,\mathbb{R}} < \tilde{s}_2(z)\} \\ &\quad - \sharp\{n \in \mathbf{I}; -\lambda_{n,\mathbb{C}} = z, \tilde{s}_2(z) < -\lambda_{n,\mathbb{R}} < \tilde{s}_1(z)\}. \end{aligned}$$

Now assume $s_1, s_2 \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$, hence \tilde{s}_i can be taken as continuous functions. Then the subset

$$\text{supp}(k_{s_1, s_2}) := \{z \in \mathbb{C}; k_{s_1, s_2}(z) \neq 0\} \subset \mathbb{C}$$

is discrete and closed because $\{\lambda_n \in \text{Im}\mathbb{H}; n \in \mathbf{I}\} \subset \text{Im}\mathbb{H}$ is also discrete and closed.

Conversely, let $s_1 \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$ and s_2 is not necessary to be continuous. If $\text{supp}(k_{s_1, s_2})$ is a discrete and closed subset of \mathbb{C} , then we can take \tilde{s}_2 to be continuous, consequently s_2 becomes continuous. Thus we obtain the following proposition.

Proposition 4.5. *Let $s_1 \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$. A map $s_2 : \mathbb{C} \rightarrow \pi_\lambda(Y_\lambda)$ which satisfies $[\text{pr}_\mathbb{C}]_\lambda \circ s_2 = \text{id}_\mathbb{C}$ is continuous if and only if $\text{supp}(k_{s_1, s_2})$ is discrete and closed.*

5 Biholomorphisms

5.1 Outline of the constructions

In this subsection we explain how to construct biholomorphisms between $X(\lambda)$ and $X(\lambda')$ for some generic $\lambda, \lambda' \in (\text{Im}\mathbb{H})_0^\mathbf{I}$. The biholomorphisms between $X(\lambda)$ and $X(\lambda')$ will be constructed if there exists a homeomorphism $\mathbf{h} : \text{Im}\mathbb{H}/\sim_\lambda \rightarrow \text{Im}\mathbb{H}/\sim_{\lambda'}$ preserving partial orders \preceq , which satisfies $[\text{pr}_\mathbb{C}]_{\lambda'} \circ \mathbf{h} = [\text{pr}_\mathbb{C}]_\lambda$.

For each continuous section $s \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$, we have an open subset

$$\begin{aligned} X(\lambda)^s &:= \mu_\lambda^{-1}(\pi_\lambda^{-1}(s(\mathbb{C}))) \\ &= p_\lambda^{-1}([\mu_\lambda]^{-1}(s(\mathbb{C}))) \subset X(\lambda), \end{aligned}$$

and it is easy to see $X(\lambda)^* = \bigcup_{s \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))} X(\lambda)^s$. In Section 5.2 the holomorphic coordinates over $X(\lambda)^s$ are constructed. By combining these holomorphic coordinates we obtain biholomorphic maps $X(\lambda)^s \rightarrow X(\lambda')^{\mathbf{h}(s)}$, then show that these glue on the intersections $X(\lambda)^{s_1} \cap X(\lambda)^{s_2}$ for all $s_1, s_2 \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$ in Section 5.3. Thus we obtain a biholomorphic map $X(\lambda)^* \rightarrow X(\lambda')^*$, which can be extended to a biholomorphic map $X(\lambda) \rightarrow X(\lambda')$.

5.2 Holomorphic coordinates on $X(\lambda)^s$

In this section we assume that $\lambda \in (\text{Im}\mathbb{H})_0^\mathbf{I}$ is generic and $\lambda_{n, \mathbb{R}} \neq 0$ for all $n \in \mathbf{I}$. We may assume the latter condition without loss of generality since there exists an isomorphism $X(\lambda) \cong X(\lambda + \underline{\eta})$ for all $\eta \in \text{Im}\mathbb{H}$ from Section 2.2.

First of all we see that there exist \mathbb{C}^\times -equivariant holomorphic functions on $X(\lambda)^{\mathbf{o}_\lambda}$, where $\mathbf{o}_\lambda \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$ is defined by $\mathbf{o}_\lambda(z) := \pi_\lambda(0, z)$. $\mathbf{I}_\lambda^\pm(\mathbf{o}_\lambda)$ are given by

$$\begin{aligned} \mathbf{I}_\lambda^+(\mathbf{o}_\lambda) &= \{n \in \mathbf{I}; \lambda_{n, \mathbb{R}} > 0\}, \\ \mathbf{I}_\lambda^-(\mathbf{o}_\lambda) &= \{n \in \mathbf{I}; \lambda_{n, \mathbb{R}} < 0\}. \end{aligned}$$

Proposition 5.1. *Let $[z + wj] \in X(\lambda)^{\mathbf{o}_\lambda}$. Then z_n is nonzero if $n \in \mathbf{I}_\lambda^+(\mathbf{o}_\lambda)$, and w_n is nonzero if $n \in \mathbf{I}_\lambda^-(\mathbf{o}_\lambda)$.*

Proof. We have $\mu_\lambda([z + wj]) \sim_\lambda (0, \mu_{\lambda, \mathbb{C}}([z + wj]))$ from the assumption $[z + wj] \in X(\lambda)^{\mathbf{o}_\lambda}$. By the injectivity of $[\mu_\lambda]$, there exists $g \in \mathbb{C}^\times$ such that $\mu_\lambda([z + wj]g) = (0, \mu_{\lambda, \mathbb{C}}([z + wj]))$. Thus we may suppose $\mu_{\lambda, \mathbb{R}}([z + wj]) = 0$, and we have $|z_n|^2 - |w_n|^2 = \lambda_{n, \mathbb{R}}$. Hence we obtain $|z_n|^2 > 0$ if $\lambda_{n, \mathbb{R}} > 0$, and $|w_n|^2 > 0$ if $\lambda_{n, \mathbb{R}} < 0$. \square

Now we consider the infinite product

$$\left(\prod_{n \in \mathbf{I}_\lambda^+(\mathbf{o}_\lambda)} \frac{z_n}{\alpha_n} \right) \left(\prod_{n \in \mathbf{I}_\lambda^-(\mathbf{o}_\lambda)} \frac{w_n}{\beta_n} \right)^{-1} \quad (1)$$

for $z + wj \in \hat{\mu}_{\Lambda, \mathbb{C}}^{-1}(0)_{\lambda_{\mathbb{R}}}$ such that $[z + wj] \in X(\lambda)^{\mathbf{o}_\lambda}$, where we take $\Lambda \in \mathbb{H}^{\mathbf{I}}$ and $\alpha = (\alpha_n)_{n \in \mathbf{I}}, \beta = (\beta_n)_{n \in \mathbf{I}} \in \mathbb{C}^{\mathbf{I}}$ to be $\Lambda_n i \bar{\Lambda}_n = \lambda_n$ and $\Lambda_n = \alpha_n + \beta_n j$. If we put $u_n := z_n - \alpha_n$ and $v_n := w_n - \beta_n$, then we can see $\sum_{n \in \mathbf{I}} |u_n|^2 < +\infty$ and $\sum_{n \in \mathbf{I}} |v_n|^2 < +\infty$. On the other hand, we can deduce

$$\sum_{n \in \mathbf{I}_\lambda^+(\mathbf{o}_\lambda)} \frac{1}{|\alpha_n|^2} < +\infty, \quad \sum_{n \in \mathbf{I}_\lambda^-(\mathbf{o}_\lambda)} \frac{1}{|\beta_n|^2} < +\infty$$

since $2|\alpha_n|^2 = |\lambda_n| + \lambda_{n, \mathbb{R}} \geq |\lambda_n|$ for $n \in \mathbf{I}_\lambda^+(\mathbf{o}_\lambda)$, and $2|\beta_n|^2 = |\lambda_n| - \lambda_{n, \mathbb{R}} \geq |\lambda_n|$ for $n \in \mathbf{I}_\lambda^-(\mathbf{o}_\lambda)$. Then the Cauchy-Schwarz inequality gives $\sum_{n \in \mathbf{I}_\lambda^+(\mathbf{o}_\lambda)} \frac{|u_n|}{|\alpha_n|} < +\infty$ and $\sum_{n \in \mathbf{I}_\lambda^-(\mathbf{o}_\lambda)} \frac{|v_n|}{|\beta_n|} < +\infty$, hence the infinite product (1) converges by the next lemma.

Lemma 5.2. *Let $x_n \in \mathbb{C} \setminus \{-1\}$ for $n = 1, 2, \dots$. If we have $\sum_{n=1}^\infty |x_n| < +\infty$, then there exists a limit $\lim_{N \rightarrow \infty} \prod_{n=1}^N (1 + x_n) \neq 0$.*

Proof. Since $1 + x_n \neq 0$, we may put $1 + x_n = e^{a_n + b_n i}$ for some $a_n, b_n \in \mathbb{R}$ such that $-\pi < b_n \leq \pi$. Then we have $\prod_{n=1}^N (1 + x_n) = e^{\sum_{n=1}^N a_n + b_n i}$, therefore it suffices to show the convergence of the series $\sum_{n=1}^\infty |a_n + b_n i|$. From the assumption $\sum_{n=1}^\infty |x_n| < +\infty$, we may suppose there exists a sufficiently large positive integer N_0 , and $|x_n| < \frac{1}{2}$ for all $n \geq N_0$. Then we have

$$a_n + b_n i = \log(1 + x_n) = \sum_{k=1}^\infty (-1)^{n-1} \frac{x_n^k}{k}$$

for every $n \geq N_0$. Consequently, we can deduce

$$|a_n + b_n i| \leq |x_n| \sum_{k=1}^\infty \frac{|x_n|^{k-1}}{k} \leq \left(\sum_{k=1}^\infty \frac{1}{k 2^{k-1}} \right) |x_n|.$$

Thus we obtain

$$\sum_{n=1}^{\infty} |a_n + b_n i| \leq \sum_{n=1}^{N_0} |a_n + b_n i| + \sum_{k=1}^{\infty} \frac{1}{k 2^{k-1}} \sum_{n=N_0}^{\infty} |x_n| < +\infty.$$

□

From Proposition 5.1 and Lemma 5.2, the value of (1) is nonzero if $[z + wj] \in X_{GIT}(\lambda)^{\mathbf{o}_\lambda}$. Moreover the function (1) is $G_\lambda^\mathbb{C}$ -invariant, consequently, it induces a smooth function $f_\lambda^{\mathbf{o}_\lambda} : X(\lambda)^{\mathbf{o}_\lambda} \rightarrow \mathbb{C}^\times$ defined by

$$f_\lambda^{\mathbf{o}_\lambda}([z + wj]) := \prod_{n \in \mathbf{I}_\lambda^+(\mathbf{o}_\lambda)} \frac{z_n}{\alpha_n} \cdot \left(\prod_{n \in \mathbf{I}_\lambda^-(\mathbf{o}_\lambda)} \frac{w_n}{\beta_n} \right)^{-1}$$

for $[z + wj] \in X_{GIT}(\lambda)^{\mathbf{o}_\lambda}$. It is easy to check that $f_\lambda^{\mathbf{o}_\lambda}$ is \mathbb{C}^\times -equivariant, in the sense $f_\lambda^{\mathbf{o}_\lambda}([z + wj]g) = g f_\lambda^{\mathbf{o}_\lambda}([z + wj])$ for all $g \in \mathbb{C}^\times$.

Proposition 5.3. *On $X(\lambda)^{\mathbf{o}_\lambda}$, the holomorphic symplectic form is given by*

$$2\omega_{\lambda, \mathbb{C}} = \frac{df_\lambda^{\mathbf{o}_\lambda}}{f_\lambda^{\mathbf{o}_\lambda}} \wedge d\mu_{\lambda, \mathbb{C}}$$

Proof. Let $\iota_\Lambda : \hat{\mu}_{\Lambda, \mathbb{C}}^{-1}(0)_{\lambda_\mathbb{R}} \rightarrow M_\Lambda$ be the embedding map, and $\pi_\Lambda : \hat{\mu}_{\Lambda, \mathbb{C}}^{-1}(0)_{\lambda_\mathbb{R}} \rightarrow X_{GIT}(\lambda)$ be the quotient map. Since $\omega_{\lambda, \mathbb{C}}$ is defined by $\pi_\Lambda^* \omega_{\lambda, \mathbb{C}} = \iota_\Lambda^* \sum_{n \in \mathbf{I}} dz_n \wedge dw_n$, we have

$$\begin{aligned} \pi_\Lambda^*(df_\lambda^{\mathbf{o}_\lambda} \wedge d\mu_{\lambda, \mathbb{C}})_{[z + wj]} &= d \left\{ \prod_{n \in \mathbf{I}_\lambda^+(\mathbf{o}_\lambda)} \frac{z_n}{\alpha_n} \cdot \left(\prod_{n \in \mathbf{I}_\lambda^-(\mathbf{o}_\lambda)} \frac{w_n}{\beta_n} \right)^{-1} \right\} \wedge (d\mu_{\lambda, \mathbb{C}})_{[z + wj]} \\ &= f_\lambda^{\mathbf{o}_\lambda}([z + wj]) \left(\sum_{n \in \mathbf{I}_\lambda^+(\mathbf{o}_\lambda)} \frac{dz_n}{z_n} \wedge d(2z_n w_n) \right. \\ &\quad \left. - \sum_{n \in \mathbf{I}_\lambda^-(\mathbf{o}_\lambda)} \frac{dw_n}{w_n} \wedge d(2z_n w_n) \right) \\ &= 2f_\lambda^{\mathbf{o}_\lambda}([z + wj]) \sum_{n \in \mathbf{I}} dz_n \wedge dw_n. \end{aligned}$$

Here we use $\mu_{\lambda, \mathbb{C}}([z + wj]) = 2z_n w_n - \lambda_{n, \mathbb{C}}$ for any $n \in \mathbf{I}$. □

The next lemma may be well-known, but we show it for the reader's convenience.

Lemma 5.4. *Let U be a complex manifold of dimension n and $f_1, \dots, f_n \in C^\infty(U)$. If $df_1 \wedge \dots \wedge df_n \in \Omega^{n,0}(U)$ and $df_1 \wedge \dots \wedge df_n|_p \neq 0$ for all $p \in U$, then $(f_1, \dots, f_n) : U \rightarrow \mathbb{C}^n$ is a local biholomorphism.*

Proof. Since $df_1 \wedge \dots \wedge df_n$ is in $\Omega^{(n,0)}(U)$ and never be zero, we have

$$df_1 \wedge \dots \wedge df_n = \partial f_1 \wedge \dots \wedge \partial f_n \neq 0.$$

Therefore $\partial f_1, \dots, \partial f_n$ becomes a basis of $(T_p^*U)^{(1,0)}$ for all $p \in U$. Since $(n-1, 1)$ -part of $df_1 \wedge \dots \wedge df_n$ vanishes, we have $\bar{\partial} f_i = 0$. Then $(f_1, \dots, f_n) : U \rightarrow \mathbb{C}^n$ is locally biholomorphic since the Jacobian is everywhere invertible because $\partial f_1 \wedge \dots \wedge \partial f_n \neq 0$. \square

From Lemma 5.4, we obtain a local holomorphic chart

$$(f_\lambda^{\mathbf{o}_\lambda}, \mu_{\lambda, \mathbb{C}}) : X(\lambda)^{\mathbf{o}_\lambda} \rightarrow \mathbb{C}^\times \times \mathbb{C}.$$

To show that $(f_\lambda^{\mathbf{o}_\lambda}, \mu_{\lambda, \mathbb{C}})$ is biholomorphic, it suffices to show that the map is bijective. We will show it later.

Next we consider \mathbb{C}^\times -equivariant holomorphic functions over $X(\lambda)^s$ for an arbitrary $s \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$.

Take a map $k : \mathbb{C} \rightarrow \mathbb{Z}$ such that $\text{supp}(k) = k^{-1}(\mathbb{Z} \setminus \{0\}) \subset \mathbb{C}$ is discrete and closed. Then denote by $\mathcal{A}(k)$ the subset of all meromorphic functions on \mathbb{C} , which consists of the meromorphic functions φ who have the limits

$$\lim_{w \rightarrow z} \varphi(w)(w - z)^{-k(z)} \in \mathbb{C}^\times$$

for all $z \in \text{supp}(k)$. Then φ is a \mathbb{C}^\times valued holomorphic function on $k^{-1}(0)$.

Now we put

$$f_\lambda^{\mathbf{o}_\lambda, \varphi}(p) := f_\lambda^{\mathbf{o}_\lambda}(p) \cdot \varphi(\mu_{\lambda, \mathbb{C}}(p))$$

for $s \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$ and $\varphi \in \mathcal{A}(k_{\mathbf{o}_\lambda, s})$, which is a \mathbb{C}^\times -valued holomorphic function on $X(\lambda)^{\mathbf{o}_\lambda} \cap X(\lambda)^s$.

Proposition 5.5. *Let $s \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$ and $\varphi \in \mathcal{A}(k_{\mathbf{o}_\lambda, s})$. Then $f_\lambda^{\mathbf{o}_\lambda, \varphi}$ extends to \mathbb{C}^\times -equivariant holomorphic map $X(\lambda)^s \rightarrow \mathbb{C}^\times$.*

Proof. Since $f_\lambda^{\mathbf{o}_\lambda, \varphi}$ can be regarded as a \mathbb{C}^\times -equivariant holomorphic map $X(\lambda)^{\mathbf{o}_\lambda} \cap X(\lambda)^s \rightarrow \mathbb{C}^\times$, it suffices to show that $f_\lambda^{\mathbf{o}_\lambda, \varphi}$ is extended to $X(\lambda)^s$ continuously.

Let $[z + wj] \in X_{GIT}(\lambda)^{\mathbf{o}_\lambda} \cap X_{GIT}(\lambda)^s$. We fix $m \in \mathbf{I}$ arbitrarily, and put $\hat{Z}_A := \prod_{n \in A} \frac{z_n}{\alpha_n}$, $\hat{W}_A := \prod_{n \in A} \frac{w_n}{\beta_n}$ for $A \subset \mathbf{I}$. First of all, the following conditions are all equivalent for all $m \in \mathbf{I}$ and $s_1, s_2 \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$;

(i) $s_1(-\lambda_{m,\mathbb{C}}) \preceq s_2(-\lambda_{m,\mathbb{C}})$, (ii) $\mathbf{I}_\lambda^+(s_1) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}} \subset \mathbf{I}_\lambda^+(s_2) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}$, (iii) $\mathbf{I}_\lambda^-(s_2) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}} \subset \mathbf{I}_\lambda^-(s_1) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}$.

Assume $s(-\lambda_{m,\mathbb{C}}) \preceq \mathbf{o}_\lambda(-\lambda_{m,\mathbb{C}})$. Then we can deduce

$$\begin{aligned}
f_\lambda^{\mathbf{o}_\lambda}([z + wj]) &= \hat{Z}_{\mathbf{I}_\lambda^+(\mathbf{o}_\lambda)} \hat{W}_{\mathbf{I}_\lambda^-(\mathbf{o}_\lambda)}^{-1} \\
&= \frac{\hat{Z}_{\mathbf{I}_\lambda^+(\mathbf{o}_\lambda) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}} \hat{Z}_{\mathbf{I}_\lambda^+(\mathbf{o}_\lambda) \setminus \mathbf{I}_{-\lambda_{m,\mathbb{C}}}}}{\hat{W}_{\mathbf{I}_\lambda^-(\mathbf{o}_\lambda) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}} \hat{W}_{\mathbf{I}_\lambda^-(\mathbf{o}_\lambda) \setminus \mathbf{I}_{-\lambda_{m,\mathbb{C}}}}} \\
&= \frac{\hat{Z}_{(\mathbf{I}_\lambda^+(\mathbf{o}_\lambda) \setminus \mathbf{I}_\lambda^+(s)) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}} \hat{Z}_{\mathbf{I}_\lambda^+(s) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}} \hat{Z}_{\mathbf{I}_\lambda^+(\mathbf{o}_\lambda) \setminus \mathbf{I}_{-\lambda_{m,\mathbb{C}}}}}{\hat{W}_{(\mathbf{I}_\lambda^-(s) \setminus \mathbf{I}_\lambda^-(\mathbf{o}_\lambda)) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}}^{-1} \hat{W}_{\mathbf{I}_\lambda^-(s) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}} \hat{W}_{\mathbf{I}_\lambda^-(\mathbf{o}_\lambda) \setminus \mathbf{I}_{-\lambda_{m,\mathbb{C}}}}} \\
&= \frac{\hat{Z}_{\mathbf{I}_\lambda^+(s) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}} \hat{Z}_{\mathbf{I}_\lambda^+(\mathbf{o}_\lambda) \setminus \mathbf{I}_{-\lambda_{m,\mathbb{C}}}}}{\hat{W}_{\mathbf{I}_\lambda^-(s) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}} \hat{W}_{\mathbf{I}_\lambda^-(\mathbf{o}_\lambda) \setminus \mathbf{I}_{-\lambda_{m,\mathbb{C}}}}} \times \prod_{n \in (\mathbf{I}_\lambda^+(\mathbf{o}_\lambda) \setminus \mathbf{I}_\lambda^+(s)) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}} \frac{z_n w_n}{\alpha_n \beta_n}.
\end{aligned}$$

Here we use $\mathbf{I}_\lambda^+(\mathbf{o}_\lambda) \setminus \mathbf{I}_\lambda^+(s) = \mathbf{I}_\lambda^-(s) \setminus \mathbf{I}_\lambda^-(\mathbf{o}_\lambda)$ for the last equality. Now we put $\zeta_{\mathbb{C}} = \mu_{\lambda,\mathbb{C}}([z + wj])$. Then we have $\zeta_{\mathbb{C}} = 2z_n w_n - \lambda_{n,\mathbb{C}}$ and $2\alpha_n \beta_n = \lambda_{n,\mathbb{C}}$, hence

$$\frac{z_n w_n}{\alpha_n \beta_n} = \frac{\zeta_{\mathbb{C}} + \lambda_{m,\mathbb{C}}}{\lambda_{m,\mathbb{C}}}$$

if $n \in \mathbf{I}_{-\lambda_{m,\mathbb{C}}}$. Thus we obtain

$$f_\lambda^{\mathbf{o}_\lambda}([z + wj]) = \left(\frac{\zeta_{\mathbb{C}} + \lambda_{m,\mathbb{C}}}{2} \right)^{-k_{\mathbf{o}_\lambda, s}(-\lambda_{m,\mathbb{C}})} \frac{\hat{Z}_{\mathbf{I}_\lambda^+(s) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}} \hat{Z}_{\mathbf{I}_\lambda^+(\mathbf{o}_\lambda) \setminus \mathbf{I}_{-\lambda_{m,\mathbb{C}}}}}{\hat{W}_{\mathbf{I}_\lambda^-(s) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}} \hat{W}_{\mathbf{I}_\lambda^-(\mathbf{o}_\lambda) \setminus \mathbf{I}_{-\lambda_{m,\mathbb{C}}}}}.$$

Since

$$\varphi(\zeta_{\mathbb{C}}) \left(\frac{\zeta_{\mathbb{C}} + \lambda_{m,\mathbb{C}}}{2} \right)^{-k_{\mathbf{o}_\lambda, s}(-\lambda_{m,\mathbb{C}})}$$

is \mathbb{C}^\times -valued holomorphic on the neighborhood of $\zeta_{\mathbb{C}} = -\lambda_{m,\mathbb{C}}$, and

$$\frac{\hat{Z}_{\mathbf{I}_\lambda^+(s) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}} \hat{Z}_{\mathbf{I}_\lambda^+(\mathbf{o}_\lambda) \setminus \mathbf{I}_{-\lambda_{m,\mathbb{C}}}}}{\hat{W}_{\mathbf{I}_\lambda^-(s) \cap \mathbf{I}_{-\lambda_{m,\mathbb{C}}}} \hat{W}_{\mathbf{I}_\lambda^-(\mathbf{o}_\lambda) \setminus \mathbf{I}_{-\lambda_{m,\mathbb{C}}}}}$$

is also \mathbb{C}^\times -valued at $\zeta_{\mathbb{C}} = -\lambda_{m,\mathbb{C}}$, then $f_\lambda^{\mathbf{o}_\lambda, \varphi}$ can be extended continuously to $\mu_\lambda^{-1}(\pi_\lambda^{-1}(s(-\lambda_{m,\mathbb{C}})))$ for each $m \in \mathbf{I}$, accordingly extended to $X(\lambda)^s$. \square

Now $(f_\lambda^{\mathbf{o}_\lambda, \varphi}, \mu_{\lambda, \mathbb{C}}) : X(\lambda)^s \rightarrow \mathbb{C}^\times \times \mathbb{C}$ is locally biholomorphic since we have

$$df_\lambda^{\mathbf{o}_\lambda, \varphi} \wedge d\mu_{\lambda, \mathbb{C}} = 2f_\lambda^{\mathbf{o}_\lambda, \varphi} \omega_{\lambda, \mathbb{C}}$$

on $X(\lambda)^s$. The above equation follows from Proposition 5.3 and the definition of $f_\lambda^{\mathbf{o}_\lambda, \varphi}$.

Proposition 5.6. *Let $s \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$ and $\varphi \in \mathcal{A}(k_{\mathbf{o}_\lambda, s})$. Then*

$$(f_\lambda^{\mathbf{o}_\lambda, \varphi}, \mu_{\lambda, \mathbb{C}}) : X(\lambda)^s \rightarrow \mathbb{C}^\times \times \mathbb{C}$$

is biholomorphic.

Proof. Since we have shown that the map is locally biholomorphic, it suffices to show that it is bijective.

First of all we show the injectivity. Let $[z + wj], [z' + w'j] \in X_{GIT}(\lambda)^s$ satisfy

$$\mu_{\lambda, \mathbb{C}}([z + wj]) = \mu_{\lambda, \mathbb{C}}([z' + w'j]), \quad (2)$$

$$f_\lambda^{\mathbf{o}_\lambda, \varphi}([z + wj]) = f_\lambda^{\mathbf{o}_\lambda, \varphi}([z' + w'j]). \quad (3)$$

Then (2) gives that $\pi_\lambda(\mu_\lambda([z + wj])) = \pi_\lambda(\mu_\lambda([z' + w'j]))$. From Proposition 4.3, there exists $g \in \mathbb{C}^\times$ such that $[z + wj]g = [z' + w'j]$. Therefore we have $[z + wj] = [z' + w'j]$ since $f_\lambda^{\mathbf{o}_\lambda, \varphi}$ is \mathbb{C}^\times -equivariant and \mathbb{C}^\times -valued, which gives $g = 1$.

Next we show the surjectivity. Take $(p, q) \in \mathbb{C}^\times \times \mathbb{C}$ arbitrarily. Fix $[z + wj] \in \mu_\lambda^{-1}(\pi_\lambda^{-1}(s(q)))$. If we put $g_0 := f_\lambda^{\mathbf{o}_\lambda, \varphi}([z + wj])$, then $[z + wj]g_0^{-1}p \in X_{GIT}(\lambda)^s$ satisfies $f_\lambda^{\mathbf{o}_\lambda, \varphi}([z + wj]g_0^{-1}p) = p$ and $\mu_{\lambda, \mathbb{C}}([z + wj]g_0^{-1}p) = \mu_{\lambda, \mathbb{C}}([z + wj]) = q$. \square

For all $s \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$, $\mathcal{A}(k_{\mathbf{o}_\lambda, s})$ is not empty from Weierstrass Theorem. If we put $\mathcal{G} := \{f : \mathbb{C} \rightarrow \mathbb{C}^\times \text{ is holomorphic}\} = \Gamma(\mathbb{C}, \mathcal{O}_\mathbb{C}^\times)$, then \mathcal{G} acts on $\mathcal{A}(k_{\mathbf{o}_\lambda, s})$ transitively and freely.

Next we consider the gluings. Take $s_1, s_2 \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$, and $\varphi_i \in \mathcal{A}(k_{\mathbf{o}_\lambda, s_i})$ for $i = 1, 2$. We put $F_\lambda^\varphi := (f_\lambda^{\mathbf{o}_\lambda, \varphi}, \mu_{\lambda, \mathbb{C}})$ and define

$$\psi_\lambda^{\varphi_2, \varphi_1} : F_\lambda^{\varphi_1}(X(\lambda)^{s_1} \cap X(\lambda)^{s_2}) \rightarrow F_\lambda^{\varphi_2}(X(\lambda)^{s_1} \cap X(\lambda)^{s_2})$$

by $\psi_\lambda^{\varphi_2, \varphi_1} := F_\lambda^{\varphi_2} \circ (F_\lambda^{\varphi_1})^{-1}$. Now we take $p \in \mathbb{C}^\times$ and $q \in \mathbb{C}$ to be $(p, q) \in F_\lambda^{\varphi_1}(X(\lambda)^{s_1} \cap X(\lambda)^{s_2})$. Since we have

$$\begin{aligned} F_\lambda^{\varphi_2} &= (f_\lambda^{\mathbf{o}_\lambda, \varphi_2}, \mu_{\lambda, \mathbb{C}}) \\ &= (f_\lambda^{\mathbf{o}_\lambda} \cdot \varphi_1(\mu_{\lambda, \mathbb{C}}) \cdot \frac{\varphi_2(\mu_{\lambda, \mathbb{C}})}{\varphi_1(\mu_{\lambda, \mathbb{C}})}, \mu_{\lambda, \mathbb{C}}) \\ &= (f_\lambda^{\mathbf{o}_\lambda, \varphi_1} \cdot \frac{\varphi_2(\mu_{\lambda, \mathbb{C}})}{\varphi_1(\mu_{\lambda, \mathbb{C}})}, \mu_{\lambda, \mathbb{C}}), \end{aligned}$$

then we can write as

$$\psi_\lambda^{\varphi_2, \varphi_1}(p, q) = (p \cdot \frac{\varphi_2(q)}{\varphi_1(q)}, q). \quad (4)$$

Consequently, we have

$$F_\lambda^{\varphi_1}(X(\lambda)^{s_1} \cap X(\lambda)^{s_2}) = F_\lambda^{\varphi_2}(X(\lambda)^{s_1} \cap X(\lambda)^{s_2}) = \mathbb{C}^\times \times k_{s_1, s_2}^{-1}(\{0\}).$$

5.3 The construction of biholomorphic maps

Recall that we have put $X(\lambda)^* = \mu_\lambda^{-1}(Y_\lambda)$. In this section we construct biholomorphisms between $X(\lambda)^*$ and $X(\lambda')^*$ for $\lambda, \lambda' \in (\mathrm{Im}\mathbb{H})_0^{\mathbf{I}}$, which satisfy appropriate conditions. First of we describe these conditions for $\lambda, \lambda' \in (\mathrm{Im}\mathbb{H})_0^{\mathbf{I}}$.

Let $\lambda, \lambda' \in (\mathrm{Im}\mathbb{H})_0^{\mathbf{I}}$ be generic. Then we denote by $\mathrm{Isom}(\lambda, \lambda')$ the set which consists of all homeomorphisms $\mathbf{h} : \mathrm{Im}\mathbb{H}/\sim_\lambda \rightarrow \mathrm{Im}\mathbb{H}/\sim_{\lambda'}$ preserving partial orders \preceq , which satisfies $[\mathrm{pr}_\mathbb{C}]_{\lambda'} \circ \mathbf{h} = [\mathrm{pr}_\mathbb{C}]_\lambda$. We can construct a \mathbb{C}^\times -equivariant biholomorphism from $X(\lambda)^*$ and $X(\lambda')^*$ which preserves holomorphic symplectic forms $\omega_{\lambda, \mathbb{C}}$ and $\omega_{\lambda', \mathbb{C}}$ as follows.

Let $\lambda, \lambda' \in (\mathrm{Im}\mathbb{H})_0^{\mathbf{I}}$ be generic, and $\mathbf{h} \in \mathrm{Isom}(\lambda, \lambda')$. Then \mathbf{h} induces a one-to-one correspondence $\Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda)) \rightarrow \Gamma(\mathbb{C}, \pi_{\lambda'}(Y_{\lambda'}))$ which we use the same symbol $\mathbf{h} : \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda)) \rightarrow \Gamma(\mathbb{C}, \pi_{\lambda'}(Y_{\lambda'}))$. We may assume that $\lambda_{n, \mathbb{R}} \neq 0$ and $\lambda'_{n, \mathbb{R}} \neq 0$ for all $n \in \mathbf{I}$ without loss of generality. Then $\mathrm{supp}(k_{\mathbf{o}_{\lambda'}, \mathbf{h}(\mathbf{o}_\lambda)})$ becomes discrete and closed from Proposition 4.5, accordingly we can take $\varphi_0 \in \mathcal{A}(k_{\mathbf{o}_{\lambda'}, \mathbf{h}(\mathbf{o}_\lambda)})$ since $\mathcal{A}(k_{\mathbf{o}_{\lambda'}, \mathbf{h}(\mathbf{o}_\lambda)})$ is not empty by Weierstrass Theorem.

To construct biholomorphisms from $X(\lambda)^*$ to $X(\lambda')^*$, it suffices to construct biholomorphisms from $X(\lambda)^s$ to $X(\lambda')^{\mathbf{h}(s)}$ and glue them since a family of open sets $\{X(\lambda)^s\}_{s \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))}$ is an open covering of $X(\lambda)^*$.

Recall that $F_\lambda^\varphi : X(\lambda)^s \rightarrow \mathbb{C}^\times \times \mathbb{C}$ is a biholomorphism for each $\varphi \in \mathcal{A}(k_{\mathbf{o}_\lambda, s})$. Now we have $k_{\mathbf{o}_\lambda, s} = k_{\mathbf{h}(\mathbf{o}_\lambda), \mathbf{h}(s)}$ since \mathbf{h} preserves the partial orders, hence $\varphi\varphi_0$ is an element of $\mathcal{A}(k_{\mathbf{o}_{\lambda'}, \mathbf{h}(s)})$. Consequently, we have a biholomorphism $F_{\lambda'}^{\varphi\varphi_0} : X(\lambda')^{\mathbf{h}(s)} \rightarrow \mathbb{C}^\times \times \mathbb{C}$, then a biholomorphism $H_{s, \varphi}(\mathbf{h}, \varphi_0) : X(\lambda)^s \rightarrow X(\lambda')^{\mathbf{h}(s)}$ is obtained by

$$H_{s, \varphi}(\mathbf{h}, \varphi_0) := (F_{\lambda'}^{\varphi\varphi_0})^{-1} \circ F_\lambda^\varphi$$

for $s \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$, $\varphi_0 \in \mathcal{A}(k_{\mathbf{o}_{\lambda'}, \mathbf{h}(\mathbf{o}_\lambda)})$ and $\varphi \in \mathcal{A}(k_{\mathbf{o}_\lambda, s})$.

Proposition 5.7. *Let $\lambda, \lambda' \in (\mathrm{Im}\mathbb{H})_0^{\mathbf{I}}$ be generic. Then $H_{s_1, \varphi_1}(\mathbf{h}, \varphi_0)$ and $H_{s_2, \varphi_2}(\mathbf{h}, \varphi_0)$ are glued on $X(\lambda)^{s_1} \cap X(\lambda)^{s_2}$ for all $\mathbf{h} \in \mathrm{Isom}(\lambda, \lambda')$, $s_1, s_2 \in \Gamma(\mathbb{C}, \pi_\lambda(Y_\lambda))$, $\varphi_0 \in \mathcal{A}(k_{\mathbf{o}_{\lambda'}, \mathbf{h}(\mathbf{o}_\lambda)})$ and $\varphi_i \in \mathcal{A}(k_{\mathbf{o}_\lambda, s_i})$.*

Proof. Recall that we have $F_\lambda^{\varphi_1} = \psi_\lambda^{\varphi_1, \varphi_2} \circ F_\lambda^{\varphi_2}$ on $U := X(\lambda)^{s_1} \cap X(\lambda)^{s_2}$ and $F_\lambda^{\varphi_1}(U) = F_\lambda^{\varphi_2}(U)$. By the definition of $H_{s_1, \varphi_1}(\mathbf{h}, \varphi_0)$, we can see

$$\begin{aligned} H_{s_1, \varphi_1}(\mathbf{h}, \varphi_0)|_U &= (F_{\lambda'}^{\varphi_1 \varphi_0})^{-1} \circ F_\lambda^{\varphi_1}|_U \\ &= (\psi_{\lambda'}^{\varphi_1 \varphi_0, \varphi_2 \varphi_0} \circ F_{\lambda'}^{\varphi_2 \varphi_0})^{-1} \circ \psi_\lambda^{\varphi_1, \varphi_2} \circ F_\lambda^{\varphi_2}|_U \\ &= (F_{\lambda'}^{\varphi_2 \varphi_0})^{-1} \circ (\psi_{\lambda'}^{\varphi_1 \varphi_0, \varphi_2 \varphi_0})^{-1} \circ \psi_\lambda^{\varphi_1, \varphi_2} \circ F_\lambda^{\varphi_2}|_U. \end{aligned}$$

For each $(p, q) \in F_\lambda^{\varphi_1}(U) = F_\lambda^{\varphi_2}(U)$, we have

$$\begin{aligned} (\psi_{\lambda'}^{\varphi_1 \varphi_0, \varphi_2 \varphi_0})^{-1} \circ \psi_\lambda^{\varphi_1, \varphi_2}(p, q) &= (\psi_{\lambda'}^{\varphi_1 \varphi_0, \varphi_2 \varphi_0})^{-1} \left(p \cdot \frac{\varphi_1(q)}{\varphi_2(q)}, q \right) \\ &= \left(p \cdot \frac{\varphi_1(q)}{\varphi_2(q)} \cdot \frac{\varphi_2(q) \varphi_0(q)}{\varphi_1(q) \varphi_0(q)}, q \right) = (p, q), \end{aligned}$$

which gives

$$\begin{aligned} H_{s_1, \varphi_1}(\mathbf{h}, \varphi_0)|_U &= (F_{\lambda'}^{\varphi_2 \varphi_0})^{-1} \circ \text{id}_{F_\lambda^{\varphi_2}(U)} \circ F_\lambda^{\varphi_2}|_U \\ &= H_{s_2, \varphi_2}(\mathbf{h}, \varphi_0)|_U. \end{aligned}$$

□

From the above proposition, we have a biholomorphism

$$H_*(\mathbf{h}, \varphi_0) : X(\lambda)^* \rightarrow X(\lambda')^*$$

for each $\mathbf{h} \in \text{Isom}(\lambda, \lambda')$ and $\varphi_0 \in \mathcal{A}(k_{\mathbf{o}_{\lambda'}, \mathbf{h}(\mathbf{o}_\lambda)})$ by gluing $H_{s, \varphi}(\mathbf{h}, \varphi_0)$.

Since the submanifold $X(\lambda) \setminus X(\lambda)^*$ is codimension 2 in $X(\lambda)$, then the above map $H_*(\mathbf{h}, \varphi_0)$ is extended to $H(\mathbf{h}, \varphi_0) : X(\lambda) \rightarrow X(\lambda')$ by Hartogs' extension theorem and we have completed the proof of Theorem 1.3.

6 Applications

The Riemannian metric on $X(\lambda)$ induced from the hyperkähler structure ω_λ becomes Ricci-flat since the holonomy group of hyper-Kähler metric is contained in $Sp(1)$. It is shown in [4] that the Riemannian metric is complete.

Put $\mathbf{I} = \mathbb{Z}_{>0}$ and define $\lambda^{(\beta)} \in (\text{Im}\mathbb{H})_0^{\mathbf{I}}$ by

$$\lambda_n^{(\beta)} := n^\beta i$$

for $\beta > 1$. Let $f_{\beta_1, \beta_2}(t) := t^{\frac{\beta_2}{\beta_1}}$ for $t \geq 0$ and $f_{\beta_1, \beta_2}(t) := t$ for $t < 0$ and $\beta_1, \beta_2 > 1$. Then we have $\mathbf{h}_{\beta_1, \beta_2} \in \text{Isom}(\lambda^{(\beta_1)}, \lambda^{(\beta_2)})$ defined by

$$\mathbf{h}_{\beta_1, \beta_2}(\pi_{\lambda^{(\beta_1)}}(t, z)) := \pi_{\lambda^{(\beta_2)}}(f_{\beta_1, \beta_2}(t), z).$$

Therefore $(X(\lambda^{(\beta_1)}), \omega_{\lambda^{(\beta_1)}, \mathbb{C}})$ is isomorphic to $(X(\lambda^{(\beta_2)}), \omega_{\lambda^{(\beta_2)}, \mathbb{C}})$ as holomorphic symplectic manifolds from Section 5.

Now we denote by g_β the Ricci-flat Kähler metric induced from the hyper-Kähler structure $\omega_{\lambda^{(\beta)}}$. According to [6], the volume $V_{g_\beta}(p_0, r)$ of the geodesic ball in $X(\lambda^{(\beta)})$ with respect to g_β of radius $r > 0$ centered at $p_0 \in X(\lambda^{(\beta)})$ satisfies

$$0 < \liminf_{r \rightarrow +\infty} \frac{V_{g_\beta}(p_0, r)}{r^{4 - \frac{2}{\beta+1}}} \leq \limsup_{r \rightarrow +\infty} \frac{V_{g_\beta}(p_0, r)}{r^{4 - \frac{2}{\beta+1}}} < +\infty.$$

Thus we have the following result by putting $\alpha = 4 - \frac{2}{\beta+1}$.

Theorem 6.1. *There exist a complex manifold of dimension 2 who has a family of complete Ricci-flat Kähler metrics $\{g_\alpha\}_{3 < \alpha < 4}$ with*

$$0 < \liminf_{r \rightarrow +\infty} \frac{V_{g_\alpha}(p_0, r)}{r^\alpha} \leq \limsup_{r \rightarrow +\infty} \frac{V_{g_\alpha}(p_0, r)}{r^\alpha} < +\infty.$$

The above argument can be generalized as follows. Let $\mathbf{I} = \mathbb{Z}_{>0}$ and take $\lambda, \lambda' \in (\text{Im}\mathbb{H})_0^{\mathbf{I}}$ to be

$$\lambda_n = a_n i, \quad \lambda'_n = a'_n i,$$

where $a_n, a'_n \in \mathbb{R}$ satisfy $a_1 < a_2 < \dots$ and $a'_1 < a'_2 < \dots$. Then there exists a homeomorphism $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(a_n) = a'_n$, and we can construct $\mathbf{h} \in \text{Isom}(\lambda, \lambda')$.

Moreover, we can consider more general settings. Let $\Delta_\lambda := \{\lambda_{n, \mathbb{C}} \in \mathbb{C}; n \in \mathbf{I}\}$ and $\Delta_{\lambda'}$ are discrete and closed subsets of \mathbb{C} . Assume $\Delta_\lambda = \Delta_{\lambda'}$, and for each $z \in \Delta_\lambda$, $F(\lambda, z) := \{\lambda_{n, \mathbb{R}} \in \mathbb{R}; \lambda_{n, \mathbb{C}} = z\}$ and $F(\lambda', z)$ are isomorphic as ordered sets. Here, the order structures on $F(\lambda, z)$ is naturally induced from \mathbb{R} . Under these assumptions, we may construct a homeomorphism $f_z : \mathbb{R} \rightarrow \mathbb{R}$ such that $f_z(F(\lambda, z)) = F(\lambda', z)$ for each $z \in \Delta_\lambda = \Delta_{\lambda'}$, then extend them to a homeomorphism $\tilde{f} : \text{Im}\mathbb{H} \rightarrow \text{Im}\mathbb{H}$ such that $\tilde{f}(t, z) = (f_z(t), z)$ for $z \in \Delta_\lambda = \Delta_{\lambda'}$. Thus we have the following result.

Theorem 6.2. *Let $\lambda, \lambda' \in (\text{Im}\mathbb{H})_0^{\mathbf{I}}$ be generic and satisfy $\Delta_\lambda = \Delta_{\lambda'}$. If $\Delta_\lambda \subset \mathbb{C}$ is discrete and closed and $F(\lambda, z) \cong F(\lambda', z)$ as ordered sets for each $z \in \Delta_\lambda$, then $X(\lambda) \cong X(\lambda')$ as holomorphic symplectic manifolds.*

Acknowledgement. The author was supported by Global COE program “The research and training center for new development in Mathematics” of Graduate School of Mathematical Sciences, the University of Tokyo.

References

- [1] Anderson, T., Kronheimer, P., LeBrun, C.: Complete Ricci-flat Kähler manifolds of infinite topological type, *Commun. Math. Phys.*, **125**, 637-642 (1989)
- [2] Gibbons, G.W., Hawking, S.W.: Gravitational multi-instantons, *Phys. Lett.*, 78B:4, 430-432 (1978)
- [3] Goto, R.: On toric hyper-Kähler manifolds given by the hyper-Kähler quotient method, Infinite Analysis, Advanced Series in Mathematical Physics, Vol. 16, 317-388 (1992)
- [4] Goto, R.: On hyper-Kähler manifolds of type A_∞ , *Geom. Funct. Anal.*, **4**, No. 4, 424-454 (1994)
- [5] Goto, R.: On hyper-Kähler manifolds of type A_∞ and D_∞ , *Commun. Math. Phys.*, **198**, 469-491 (1998)
- [6] Hattori, K.: The volume growth of hyper-Kähler manifolds of type A_∞ , *J. of Geometric Analysis*, to appear
- [7] Hawking, S.W.: Gravitational Instantons, *Phys. Lett.*, **A60**, 81 (1977)
- [8] Hitchin, N.J., Karlhede, A., Lindström, U., Roček, M.: Hyper-Kähler metrics and supersymmetry, *Comm. Math. Phys.*, **108**, (4), 535-589 (1987)
- [9] Konno, H.: Variation of toric hyperKähler manifolds, *Int. J. Math.*, **14**, 289-311 (2003)
- [10] Kronheimer, P.B.: The construction of ALE spaces as hyper-Kähler quotients, *Journal of Diff. Geom.*, **29**, 665-683 (1989)
- [11] LeBrun, C.: Complete Ricci-flat Kähler metrics on \mathbb{C}^n need not be flat, *Proc. Sympos. Pure Math.*, **52**, Part 2, Amer. Math. Soc., Providence, RI, 297-304 (1991)
- [12] Newman, E., Tamburino, L., Unti, T.: Empty-Space Generalization of the Schwarzschild Metric, *J. Math. Phys.*, **4**, 915 (1963)
- [13] Taub, A.H.: Empty Space-Times Admitting a Three Parameter Group of Motions, *Ann. Math.*, **53**, No. 3, 472-490 (1951)